Negations of Quantifiers

Lecture 10
Section 3.2

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Outline

Negations of Quantified Statements

- What would it take to make the statement “Everybody loves Raymond” false?
- What would it take to make the statement “Somebody loves Raymond” false?
- What would it take to make the statement “Nobody loves Raymond” false?

Negations of Universal Statements

- The negation of the statement

$$\forall x \in S, P(x)$$

  is the statement

$$\exists x \in S, \sim P(x).$$

- If $\forall x \in \mathbb{R}, x^2 > 100$ is false, then $\exists x \in \mathbb{R}, x^2 \leq 100$ is true.
- What is the negation of the statement

$$\forall x \in \varnothing, x^2 > 100.$$

Negations of Universal Statements

- The negation of the statement

$$\exists x \in S, P(x)$$

is the statement

$$\forall x \in S, \sim P(x).$$

- If "$\exists x \in \mathbb{R}, x^2 < 0$" is false, then "$\forall x \in \mathbb{R}, x^2 \geq 0$" is true.
- What is the negation of the statement

$$\exists x \in \varnothing, x^2 > 100.$$

Negations and DeMorgan's Laws

- Let the domain be $D = \{x_1, x_2, \ldots, x_n\}$.
- The statement $\forall x \in D, P(x)$ is equivalent to

$$P(x_1) \wedge P(x_2) \wedge \cdots \wedge P(x_n).$$

- By DeMorgan's Law, its negation is

$$\sim P(x_1) \vee \sim P(x_2) \vee \cdots \vee \sim P(x_n),$$

  which is equivalent to $\exists x \in D, \sim P(x)$.

Negations and DeMorgan's Laws

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which is equivalent to $\forall x \in D, \sim P(x)$.

Example

- Are these statements equivalent?
  - “Any investment plan is not right for all investors.”
  - “There is no investment plan that is right for all investors.”

The Word "Any"

- We should avoid using the word "any" when writing quantified statements.
- The meaning of "any" is ambiguous.
- "You can't put any person in that position and expect him to perform well."
- Instead, use "all," "some," "none," or other such words.

Outline

Negation of Conditionals

- The negation of

  $$\forall x \in S, P(x) \to Q(x)$$

  is the statement

  $$\exists x \in S, \sim (P(x) \to Q(x)),$$

  which is equivalent to

  $$\exists x \in S, P(x) \wedge \sim Q(x).$$

Crows and Black Things

- Consider the statement "All crows are black."
- Let $C(x)$ be the predicate "x is a crow."
- Let $B(x)$ be the predicate "x is black."
- The statement can be written formally as

$$\forall x, C(x) \rightarrow B(x)$$

  or simply

$$C(x) \Rightarrow B(x).$$

- What would constitute statistical evidence in support of this statement?

Crows and Black Things

- The statement is logically equivalent to

$$\forall x, \sim B(x) \rightarrow \sim C(x)$$

  or simply

$$\sim B(x) \Rightarrow \sim C(x).$$

- What would constitute statistical evidence in support of this statement?
- Can you show that all crows are black without ever looking at a crow?

Outline

Necessary Conditions

- To say that P is necessary for Q means

$$\sim P \to \sim Q,$$

which is equivalent to

$$Q \to P.$$

- Write the following statements as conditionals.
  - "To be a good man and a good citizen, it is necessary that you pay your taxes."
  - "To get a good job, you need a good education and know the right person."

Sufficient Conditions

- To say that P is sufficient for Q means

$$P \to Q.$$

- Thus, "P is necessary for Q" and "P is sufficent for Q" are converses of each other.
- Write the following statements as conditionals.
  - "To be a good man and a good citizen, it is sufficient that you pay your taxes."
  - "Getting a good education or knowing the right person is sufficient to get a good job."

Necessary and Sufficient Conditions

- To say that P is necessary and sufficient for Q means

$$P \leftrightarrow Q.$$

- Write the following statements as conditionals.
  - “For n to be a multiple of 6, it is necessary and sufficient that n be a multiple of 2 and a multiple of 3.”
  - “For n to be a multiple of 8, it is necessary, but not sufficient, that n be a multiple of 4.”
  - “For n to be a multiple of 4, it is sufficient, but not necessary, that n be a multiple of 8.”

Outline

Assignment

Assignment

- Read Section 3.2, pages 108 - 115.
- Exercises 1, 3, 4, 9, 10, 17, 19, 20, 25, 37, 40, 43, page 115.